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Approximation of SαS Lévy processes in L_p norm

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Abstract

We determine the weak asymptotic behavior of linear and Kolmogorov widths of the SαS Lévy process in the Banach spaces L_p, $p \in [1, \infty)$ for $\alpha \in (0, 2)$. This complements earlier work by Maiorov and Wasilkowski, who treated the case $\alpha = 2$, i.e., the Wiener process.

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1. Result

Let $\alpha \in (0, 2]$. Recall that a real-valued random variable ξ is called symmetric α-stable (SαS) iff for the characteristic function we have

$$\hat{\xi}(\lambda) = \mathbb{E} \exp\{i\lambda\xi\} = \exp\{-|\lambda|^\alpha |\sigma^\alpha\}$$

for some $\sigma \geqslant 0$. A real-valued stochastic process $X = (X_t)_{t\in[0,1]}$ is called an SαS Lévy process iff

(i) $\sum_{i=1}^n \beta_i X_{t_i}$ is an SαS variable for any $n \in \mathbb{N}$, $t_i \in [0, 1]$, $\beta_i \in \mathbb{R}$.
(ii) $X_0 = 0$ a.s., and X has independent increments.
(iii) $(X_{ct})_{t\in[0,1/c]} \stackrel{d}{=} c^{1/\alpha}(X_t)_{t\in[0,1/c]}$ for any $c \geqslant 1$.
(iv) X has a.s. cádlàg trajectories, that is, the paths of X are a.s. continuous from the right and convergent from the left in any point.

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(Note that an S2S Lévy motion is just an ordinary Wiener process.) Due to the last property, we may and will regard X also as a random variable with values in the Banach space L_p, where $p \in [1, \infty)$. This variable turns out to be SαS again, i.e., for any continuous linear functional $\varphi : L_p \to \mathbb{R}$ the induced real-valued random variable is SαS.

We are interested in the question how good we may approximate X in L_p norm. More generally, assume that Y is a random variable in a Banach space E such that $\mathbb{E}\,\|Y\|^q < \infty$ for some $q > 0$. Then set, for $n \in \mathbb{N}$,

$$a_n(Y, E, q) := \inf\left\{\left(\mathbb{E}\,\|Y - T_n(Y)\|^q\right)^{\frac{1}{q}} \ : \ T_n : E \to E \text{ linear, rk } T_n < n\right\}.$$

These numbers are called nth *linear widths* of Y. Moreover, denoting by $Q_N : E \to E/N$ the quotient mapping to a closed linear subspace N, we set

$$d_n(Y, E, q) := \inf\left\{\left(\mathbb{E}\,\|Q_N(Y)\|^q\right)^{\frac{1}{q}} \ : \ N \subseteq E, \dim\, N < n\right\}$$

the nth *Kolmogorov width* of Y. Note $\|Q_N x\| = d(x, N) := \inf_{y\in N}\, \|x - y\|_E$. It is not hard to see that, if Y is Radon (i.e., there are compact subsets $K_n \subseteq E$ such that $\mathbb{P}\,(Y \notin K_n) \to 0$), then

$$d_n(Y, E, q) = \inf_{y_n}\, \left(\mathbb{E}\,\|Y - Y_n\|^q\right)^{\frac{1}{q}}, \tag{1}$$

where the infimum runs over all E-valued random variables Y_n such that there exists an n-dimensional subspace N with $Y_n \in N$ a.s.

Thus, the Kolmogorov widths describe how well Y may be approximated using some n-dimensional random variable. Furthermore, we define

$$r_n(Y, E, q) := \inf\left\{\left(\mathbb{E}\,\|Y - Y_n\|^q\right)^{\frac{1}{q}} \ : \ |\text{supp}\, Y_n| \leqslant 2^n\right\}.$$

This is referred to as nth (*dyadic*) *quantization number*.

If E has the approximation property, then all three sequences tend to zero iff Y is Radon and share some algebraic properties. Our aim is to determine their speed of convergence in weak asymptotics for the case of the SαS Lévy motion X in L_p spaces. (We note that in this case, we have for $\alpha < 2$ that $\mathbb{E}\,\|X\|_{L_p}^q < \infty$ iff $q < \alpha$, while for $\alpha = 2$, $\mathbb{E}\,\|X\|_{L_p}^q < \infty$ for all $q < \infty$; see [9, Chapters 3.1, 5.2].) We will only consider weak asymptotics: for two sequences a_n, b_n of real numbers, let us write $a_n \preceq b_n$ iff $\lim_n a_n/b_n < \infty$, and $a_n \asymp b_n$ iff $a_n \preceq b_n \preceq a_n$. In [5,12–14], the weak asymptotics for the case $\alpha = 2$ (i.e., the Wiener process) were determined. The results may be summarized as follows:

Theorem 1. *Let W be a Wiener process and $p \in [1, \infty]$, $q > 0$. If $p < \infty$ then*

$$a_n(W, L_p, q) \asymp d_n(W, L_p, q) \asymp r_n(W, L_p, q) \asymp n^{-1/2}.$$

In the case $p = \infty$ it holds that

$$d_n(W, L_\infty, q) \asymp r_n(W, L_\infty, q) \asymp n^{-1/2},$$

while

$$a_n(W, L_\infty, q) \asymp n^{-1/2}(\log n)^{1/2}.$$

We shall prove corresponding results for the case $\alpha < 2$. For a_n and d_n, we can state an almost complete result about weak asymptotics:

Theorem 2. *Let* $\alpha \in (0, 2)$ *and* X *be an* SαS *Lévy motion. Then for* $0 < q < \alpha$ *and* $p \in [1, \infty)$ *it holds that*

$$a_n(X, L_p, q) \asymp d_n(X, L_p, q) \asymp \begin{cases} n^{-1/\alpha}, & p < \alpha, \\ n^{-1/p}, & \alpha < p \leqslant 2, \\ n^{-1/2}, & p > 2. \end{cases}$$

In the case $p = \alpha$, *we have the bounds*

$$C^{-1} n^{-1/\alpha} \leqslant d_n(X, L_\alpha, q) \leqslant a_n(X, L_\alpha, q) \leqslant C n^{-1/\alpha} (\log n)^{1/\alpha}.$$

Note that in the case $p = \alpha$, there is a logarithmic gap in our estimates. It is interesting to note that the approximation error is of *better* order than for the Wiener process if $p < 2$, and never worse, although the Wiener process has much better pathwise properties (continuity, etc.). For quantization numbers, we have two-sided estimates only in the case $p < \alpha$:

Theorem 3. *Let* $\alpha \in (0, 2)$ *and* X *be a* SαS *Lévy motion*, $0 < q < \alpha$ *and* $p \in [1, \infty)$. *If* $p < \alpha$, *then*

$$r_n(X, L_p, q) \asymp n^{-1/\alpha}.$$

Further,

$$r_n(X, L_p, q) \preceq \begin{cases} n^{-1/p}, & \alpha < p \leqslant 2, \\ n^{-1/2}, & p > 2. \end{cases}$$

We note in passing that for the case $p = \infty$, X is not a Radon random element of L_∞, and one can easily conclude that in this case, neither of the defined sequences tends to zero.

It is also instructive to compare the results of Theorem 2 with the small deviation probabilities of the process. If Y is a random variable in a Banach space E, we set

$$\varphi(Y, E, \varepsilon) := -\log \left(\mathbb{P} \left(\|Y\| > \varepsilon \right) \right).$$

It is well-known, see e.g., [11], that for the SαS Lévy motion X we have

$$\varphi(X, L_p, \varepsilon) \asymp \varepsilon^{-\alpha}, \quad \varepsilon \to 0. \tag{2}$$

We recall that for *Gaussian* processes the connection between φ, r_n, d_n and a_n is very close; in the polynomial case (and in L_p norm, $p \in (1, \infty)$), all three sequences *always* have the same weak asymptotics, which is also the weak asymptotic of the pseudo-inverse of φ,

$$b_n(X, E) := \inf \left\{ \varepsilon > 0 \ : \ \varphi(X, E, \varepsilon) \leqslant n \right\}.$$

(We refer to [4,5,10] for more information about this connection.) By comparing Theorem 2 and (2), we recognize that such a connection is not true in general for stable processes; in contrast, d_n and b_n may differ by polynomial factors, even in the Hilbert space (L_2) case.

2. Tools

We shall need a number of technical tools in order to prove our results. The first such tool is a straightforward generalization of the arguments provided e.g. in [9], pp. 139–141, and treats the ℓ_p sums of norms of i.i.d. stable random elements of Banach spaces. Throughout the rest of the article, we will assume always that $\alpha \in (0, 2)$.

Proposition 4. *Let X_i, $i \leqslant n$, denote a sequence of i.i.d. SαS variables in a Banach space E. Then, for any* $0 < q < \alpha$, $p \in [1, \infty]$,

$$\left(\mathbb{E}\,\| (\|X_i\|_E)_{i=1}^m \|_{\ell_p^m}^q\right)^{\frac{1}{q}} \leqslant K_{\alpha,p,q} \cdot \begin{cases} m^{1/\alpha}\|X_1\|_q, & p < \alpha, \\ m^{1/p}\|X_1\|_q, & p > \alpha, \\ \|X_1\|_q m^{1/\alpha}[\log(1+m)]^{1/\alpha}, & p = \alpha. \end{cases}$$

Here, $K_{\alpha,p,q}$ depends on α, p, q solely.

Our next tool is an estimate between b_n and r_n, valid for a more general class of random variables than symmetric stable ones: let us say that a random variable X in a Banach space E has the *Anderson property* iff

$$\mathbb{P}(\|X - x\|_E > t) \geqslant \mathbb{P}(\|X\|_E > t) \quad \forall x \in E, \quad t > 0. \tag{A}$$

It is well-known that centered Gaussian elements have the Anderson property; from this and the fact that SαS elements can be represented by a mixture of centered Gaussian elements, we easily infer that SαS processes also have the Anderson property. The following inequality was essentially proved in [5].

Lemma 5. *Let X be a random variable in E with the Anderson property* (A) *and assume that* $\mathbb{E}\,\|X\|^q < \infty$. *Then*

$$b_{2n}(X, E) \leqslant c_q r_n(X, E, q).$$

We will also need an inequality similar in spirit to an inequality of Carl for entropy and approximation numbers, see [2, Theorem 3.1.1]; the result is taken from [3] and proved in Appendix A.

Theorem 6. *Let $r, q \in (0, \infty)$ be arbitrary with $q > r$, and let $\sigma > 0$. There exists a constant $C_{\sigma,q,r} > 0$ such that for any Banach space E, any Radon random variable Y in E with $\mathbb{E}\,\|Y\|^q < \infty$ and any $n \in \mathbb{N}$ the estimate*

$$\sup_{k \leqslant n} k^\sigma \cdot r_k(Y, E, r) \leqslant C_{\sigma,p,r} \cdot \sup_{k \leqslant n} k^\sigma \cdot d_k(Y, E, q)$$

is valid.

This result is no longer true for $r = q$, as simple examples show (e.g., [8, Example 6.4]).

Note that due to (1), for any Radon variable Y we have $d_n(Y, E, q) \leqslant a_n(Y, E, q)$. As a consequence of these estimates, we find:

Corollary 7. *Let Y be a Radon random variable in a Banach space E with the Anderson property* (A) *and* $\mathbb{E}\,\|Y\|^q < \infty$ *for some* $q > 0$, *and let* $\sigma > 0$, $r \in (0, q)$.
(a) *We have the implications*

$$a_n(Y, E, q) \preceq n^{-\sigma} \Rightarrow d_n(Y, E, q) \preceq n^{-\sigma} \\ \Rightarrow r_n(Y, E, r) \preceq n^{-\sigma} \Rightarrow b_n(Y, E) \preceq n^{-\sigma}.$$

(b) *Assume that* $d_n(Y, E, q) \preceq n^{-\sigma}$ *while* $r_n(Y, E, r) \succeq n^{-\sigma}$. *Then*

$$d_n(Y, E, q) \asymp r_n(Y, E, r) \asymp n^{-\sigma}.$$

(c) *If* $a_n(Y, E, q) \preceq n^{-\sigma}$ *and* $b_n(Y, E) \succeq n^{-\sigma}$, *it follows that*

$$a_n(Y, E, q) \asymp d_n(Y, E, q) \asymp r_n(Y, E, r) \asymp b_n(Y, E) \asymp n^{-\sigma}.$$

Proof. Part (a) is straightforward, and part (c) follows immediately from parts (a), (b) and Lemma 5. Hence, part (b) is the interesting conclusion. We will use an argument due to Carl (cf. [1, p. 106]). We know that $d_n(Y, E, q) \leqslant c_1 n^{-\sigma}$ while $n^{-\sigma} \leqslant c_2 r_n(Y, E, r)$. Applying Theorem 6 for the exponent 2σ, we infer the following inequalities for any $n, m \in \mathbb{N}$:

$$\begin{aligned}(mn)^\sigma &\leqslant c_2 (mn)^{2\sigma} r_{mn}(Y, E, r) \\ &\leqslant c_2 \sup_{k \leqslant mn} k^{2\sigma} r_k(Y, E, r) \\ &\leqslant c_3 \sup_{k \leqslant mn} k^{2\sigma} d_k(Y, E, r) \\ &\leqslant c_3 \sup_{k \leqslant n} k^{2\sigma} d_k(Y, E, r) + c_3 \sup_{n < k \leqslant mn} k^{2\sigma} d_k(Y, E, r) \\ &\leqslant c_4 n^\sigma + c_5 (mn)^{2\sigma} d_n(Y, E, r),\end{aligned}$$

hence

$$d_n(Y, E, r) \geqslant c_5 (mn)^{-\sigma} \cdot \left(1 - \frac{c_4}{m^\sigma}\right).$$

If we choose $m = \lfloor (2c_4)^{1/\sigma} \rfloor + 1$, it follows that $d_n(Y, E, r) \geqslant c_6 n^{-\sigma}$ for all $n \in \mathbb{N}$. □

Further, we will employ the concept of spectral measures and parameters. Recall that for a Radon SαS element Y in a Banach space E there is a measure m on E such that for any $f \in E^*$ (the topological dual) we have

$$\mathbb{E}\, e^{if(Y)} = \exp\left\{-1/2 \int_E |f(x)|^p \,\mathrm{d}m(x)\right\}.$$

This measure m need not be unique; however, the quantity

$$\sigma_\alpha(Y) := \left(\int_E \|x\|^\alpha \,\mathrm{d}m(x)\right)^{1/\alpha}$$

is finite and independent of the special choice of m. In the case of an SαS Lévy motion, a possible choice for the spectral measure is the distribution of the simple jump process defined as follows:

Let U be a uniform variable on [0, 1], and define $\xi_t := \mathbb{1}_{U \leqslant t}$. Denote by m the distribution of ξ, regarded as a random element in $L_p([0, 1])$. Then m is a spectral measure for an SαS Lévy motion X. SαS variables and their spectral measures behave nicely under bounded linear mappings.

Lemma 8. *If Y is SαS on the Banach space E with a spectral measure m and $A : E \to F$ is a bounded linear operator between Banach spaces, then $A(Y)$ is SαS on F with spectral measure $m \circ A^{-1}$. In particular,*

$$\sigma_\alpha(A(Y)) = \left(\int_E \|A(x)\|^\alpha \, \mathrm{d}\mu(x)\right)^{1/\alpha}.$$

Proof. For any $f \in F^*$ we have

$$\begin{aligned}\mathbb{E}\, e^{if(A(Y))} = \mathbb{E}\, e^{i(A^* f)(Y)} &= \exp\left\{-1/2 \int_E |A^* f(x)|^\alpha \, \mathrm{d}m(x)\right\} \\ &= \exp\left\{-1/2 \int_F |f(y)|^\alpha \, \mathrm{d}m \circ A^{-1}(y)\right\}. \qquad \square\end{aligned}$$

We shall combine this with the following remarkable fact, which follows from Theorem 9.27 of [9] and Proposition 11.11 from [6]:

Theorem 9. *Assume that $p \in [1, \infty)$ satisfies $p > \alpha$, and that E is a quotient space of $L_p([0, 1])$ (i.e., of the form $E = L_p([0, 1])/N$, where N is a closed linear subspace of $L_p[0, 1]$). Then, for any SαS variable Y in E and any $q < \alpha$, we have*

$$\left(\mathbb{E}\, \|Y\|_E^q\right)^{\frac{1}{q}} \leqslant C_{p,q,\alpha} \cdot \sigma_\alpha(Y),$$

where $C_{p,q,\alpha}$ depends on p, q, α solely.

We mention that the reverse inequality,

$$\left(\mathbb{E}\, \|Y\|_E^q\right)^{\frac{1}{q}} \geqslant c_{p,q,\alpha} \cdot \sigma_\alpha(Y),$$

is valid without any assumption on E.

Combining these results, we arrive at:

Corollary 10. *Let Y be SαS in L_p, $p \in [1, \infty)$, where $p > \alpha > q > 0$. If m is a spectral measure of Y and Z is a random variable distributed according to $m/(m(L_p))$, then*

$$a_n(Y, L_p, q) \asymp a_n(Z, L_p, \alpha)$$

and

$$d_n(Y, L_p, q) \asymp d_n(Z, L_p, \alpha).$$

Proof. We will only give the upper bound, the lower bound being similar. Regard for instance d_n. Let N be a subspace of L_p such that

$$\left(\mathbb{E}\, \|Q_N Z\|^\alpha\right)^{1/\alpha} \leqslant 2 d_n(Z, L_p, \alpha).$$

By Lemma 8 we know that $Q_N Y$ is SαS in $L_p[0,1]/N$, and that

$$\sigma_\alpha(Q_N Y) = m(L_p)\left(\mathbb{E}\,\|Q_N Z\|^\alpha\right)^{1/\alpha}.$$

Theorem 9 applies, and we derive that

$$\left(\mathbb{E}\,\|Q_N Y\|^q\right)^{1/q} \leqslant 2C_{p,q,\alpha} d_n(Z, L_p, \alpha).$$

This implies trivially that

$$d_n(Z, L_p, q) \leqslant 2C_{p,q,\alpha} d_n(Z, L_p, \alpha). \qquad \square$$

Lastly, we quote two classical results from the theory of n-widths. For a bounded linear operator $u : E \to F$ between Banach spaces, denote

$$d_n(u) := \inf_N \sup_{\|x\|\leqslant 1} \inf_{y \in N} \|u(x) - y\|,$$

where the leftmost inf runs over all n-dimensional subspaces $N \subseteq F$. These are the classical Kolmogorov n-widths of an operator.

The following result is due to Gluskin, see [7]:

Theorem 11. *Let $p \in [1, \infty)$ and denote by $\mathrm{i}^m_{1,p} : \ell^m_1 \to \ell^m_p$ the identity mapping. There is $c_p > 0$ such that for $n < c_p m$ we have*

$$d_n(\mathrm{i}^m_{1,p}) \asymp \begin{cases} 1, & p \leqslant 2, \\ n^{1/p-1/2}, & p > 2. \end{cases}$$

Lastly, we need a classical estimate for linear widths. The linear widths of a precompact subset B of a Banach space E are defined as

$$a_n(B, E) := \inf\left\{ \sup_{x \in B} \|x - v_n(x)\| \ :\ v_n : E \to E,\ \mathrm{rk}(v_n) \leqslant n \right\}.$$

Then, as reported e.g. in [15, Theorem VII.1.1.1], we have:

Theorem 12. *Let I be the integral operator, $I(f)(t) := \int_0^t f(s)\,\mathrm{d}(s)$, and set $B_{1,p} := \{I(f) : f \in L_1, \|f\|_{L_1} \leqslant 1\} \subseteq L_p$. Then*

$$a_n(B_{1,p}, L_p) \asymp \begin{cases} n^{-1/p}, & p \leqslant 2, \\ n^{-1/2}, & p > 2. \end{cases}$$

3. Proofs

Proof of Theorem 2, upper bounds for $p \leqslant 2$. For the upper bound, we infer from (1) that $d_n \leqslant a_n$ always, hence we only have to consider $a_n(X, L_p, q)$. We use the simplest equidistant approximation scheme. For $m \in \mathbb{N}$, denote $t_i := i/m$ for $i \leqslant m$, and set

$$\hat{X}^{(m)}_t := X_t - X_{t_i}, \quad t \in [t_i, t_{i+1}).$$

Thus, $\hat{X}_t^{(m)}$ is a sequence of m independent standard Lévy processes with lifetime $1/m$, starting sequentially at the times t_i. Secondly, denote

$$\bar{X}_t^{(m)} := X_{t_i}, \quad t \in [t_i, t_{i+1}].$$

Hence, for any m we have $X - \bar{X}^{(m)} = \hat{X}^{(m)}$. We would like to take $\bar{X}^{(m)}$ as an approximating element for X. A slight drawback is that $\bar{X}$ is not of the form required by the definition of the linear widths, since the coordinate functionals are not well-defined on L_p. However, we can easily find a workaround by introducing, for $\delta > 0$, an approximating operator

$$u^{m,\delta} : L_p \to L_p, \quad u^{m,\delta}(f)(t) := \frac{1}{\delta} \int_{t_i}^{t_i+\delta} f(s)\, \mathrm{d}s, \quad t \in (t_i, t_{i+1}].$$

This is obviously bounded from L_p to L_p, and since by the scaling property and stationarity of increments we have

$$\begin{aligned} X_{t_i} - \frac{1}{\delta} \int_{t_i}^{t_i+\delta} X_s \,\mathrm{d}s &= \delta^{-1} \int_{t_i}^{t_i+\delta} (X_s - X_{t_i})\, \mathrm{d}s \\ &\stackrel{d}{=} \delta^{1/\alpha} \int_0^1 X_r \,\mathrm{d}r, \end{aligned}$$

we easily conclude that, for fixed m,

$$\inf_{\delta>0} \mathbb{E}\, \|\bar{X}^{(m)} - u^{m,\delta}(X)\|_{L_p}^q \leqslant \inf_{\delta>0} \mathbb{E} \sup_i \left| X_{t_i} - \frac{1}{\delta} \int_{t_i}^{t_i+\delta} X_s \right|^q = 0,$$

and hence

$$a_m(X)^q \leqslant \inf_{\delta>0} \left(\mathbb{E}\, \|X - u^{m,\delta}(X)\|_{L_p}^q \right)^{\frac{1}{q}} \leqslant \max\{1, 2^{\frac{1}{q}}\} \cdot \left(\mathbb{E}\, \|\hat{X}^{(m)}\|_{L_p}^q \right)^{\frac{1}{q}}. \tag{3}$$

Our task is now to estimate the last expectation. To this end, we note that

$$\begin{aligned} \|\hat{X}^{(m)}\|_{L_p[0,1]} &\stackrel{d}{=} \| \left(\|X^{[i]}\|_{L_p[0,1/m]} \right)_{i=1}^m \|_{\ell_p^m} \\ &\stackrel{d}{=} m^{-1/p-1/\alpha} \cdot \| \left(\|X^{[i]}\|_{L_p[0,1]} \right)_{i=1}^m \|_{\ell_p^m}, \end{aligned}$$

where $(X^{[1]}, \ldots, X^{[m]})$ is a sequence of independent SαS Lévy processes on [0, 1]. Now we apply Proposition 4 to see that

$$\left(\mathbb{E}\, \|\hat{X}^{(m)}\|_{L_p[0,1]}^q \right)^{\frac{1}{q}} \leqslant K \cdot \begin{cases} m^{1/\alpha}, & p < \alpha, \\ m^{1/p}, & p > \alpha, \\ m^{1/\alpha}[\log(1+m)]^{1/\alpha}, & p = \alpha. \end{cases}$$

Inserting this into estimate (3) reveals the upper bounds for a_m in the case $p \leqslant 2$. □

Proof of Theorem 2, upper bounds for $p > 2$. By Corollary 10, we have to find estimates only for the numbers $a_n(\xi, L_p, \alpha)$, $d_n(\xi, L_p, \alpha)$, where $\xi_t := \mathbb{1}_{U \leqslant t}$ with U uniformly distributed on [0, 1]. Again, upper bounds have to be established only for a_n. Note that obviously, for any bounded subset $B \subseteq E$ of a Banach space E, we have $a_n(B, E) = a_n(\overline{B}, E)$, where $\overline{B}$ denotes the closure of B in E. Now, every path of ξ lies within the closure of $B_{1,p}$ in L_p. Indeed, for any $t > 0$ we may define a sequence

$$f_n(s) := \begin{cases} 0, & s < t - 1/n, \\ n, & s \in [t - 1/n, t], \\ 0, & s > t. \end{cases}$$

We note that $\|f_n\|_{L_1} = 1$ and that $I(f_n)$ tends to $\mathbb{1}_{s \leqslant t}$ in L_p norm. Hence, for any outcome of U, the path $\mathbb{1}_{U \leqslant t}$ is in $\overline{B_{1,p}}$, and thus, for a suitable $v_n : L_p \to L_p$ of rank at most n, we infer that

$$\left(\mathbb{E}\,\|\xi - v_n(\xi)\|_{L_p}^{\alpha}\right)^{1/\alpha} \leqslant \sup_s \|\mathbb{1}_{s \leqslant t} - v_n(\mathbb{1}_{s \leqslant t})\|_{L_p} \leqslant 2a_n(\overline{B_{1,p}}, L_p) = 2a_n(B_{1,p}, L_p).$$

From Theorem 12 we deduce the desired upper estimates for $a_n(\xi, L_p, \alpha)$. □

Before we turn to lower bounds, some further preparations are in order. Let us introduce a projection in L_p: given $k \in \mathbb{N}$, denote σ_k the σ-algebra over [0, 1] generated by the intervals $[(i-1)/k, i/k]$, and define, for any $f \in L_p$,

$$P^k(f) := \mathbb{E}\,(f|\sigma_k).$$

By Jensen's inequality for conditional expectations, we know that $P^k : L_p \to L_p$ is a projection of norm 1. Furthermore, its image $L_p^{(k)} := P^k(L_p)$ is isomorphic to the sequence space ℓ_p^k; to be more precise, the isomorphism $\tau_k : \ell_p^k \to L_p^{(k)}$, defined by

$$\tau_k((x_i)_{i \leqslant k}) := \sum_{i=1}^{k} x_i \cdot \mathbb{1}_{[(i-1)/k, i/k]},$$

satisfies

$$\|\tau_k((x_i)_{i \leqslant k})\|_{L_p} = k^{-1/p} \cdot \|(x_i)_{i \leqslant k}\|_{\ell_p^k} \tag{4}$$

for any $(x_i)_{i \leqslant k} \in \ell_p^k$. Further, we mention a useful property of d_n: if X is a random element in the Banach space E and if $u : E \to F$ is a bounded linear operator, then $d_n(u(X), F, q) \leqslant \|u\| \cdot d_n(X, E, q)$.

A further auxiliary lemma is a technical generalization of Theorem 11. Recall the notion $d(x, N) := \inf_{y \in N} \|x - y\|$.

Lemma 13. *For any $p \in [1, \infty)$, there are $c_1(p), c_2(p), c_3(p) > 0$ such that, for any $m \in \mathbb{N}$ and any subspace $N \subseteq \ell_p^m$ of dimension at most $c_1(p) \cdot m$ there exist distinct indices $i_1, \ldots, i_k \in \{1, \ldots, m\}$ with $k \geqslant c_2(p) \cdot m$ and such that for any $j \leqslant k$ we have*

$$d(e_{i_j}, N) \geqslant \begin{cases} c_3(p), & p \leqslant 2, \\ c_3(p) m^{1/p - 1/2}, & p \geqslant 2. \end{cases}$$

Proof. Denote

$$\alpha_m := \begin{cases} 1, & p \leqslant 2, \\ m^{1/p - 1/2}, & p \geqslant 2. \end{cases}$$

Set $c_1(p) := \min\{1/2, c_p\}$, where c_p is the constant from Theorem 11. We infer that for any subspace $N \subseteq \ell_p^m$ such that $n := \dim\ N \leqslant c_1(p)m$ there is some index i_1 such that $d(i_1, N) \geqslant C_p\alpha_m$. Since the assertion of the lemma becomes stronger when N is enlarged, we may assume without loss of generality that $(c_1(p)/2)m \leqslant n \leqslant c_1(p)m$. Consider now the projection

$$P_{i_1} : \ell_p^m \to \ell_p^{m-1}, \quad (x_1, \dots, x_m) \mapsto (x_1, \dots, x_{i_1-1}, x_{i_1+1}, \dots, x_m)$$

and $N_1 := P_{i_1}(N) \subseteq \ell_p^{m-1}$. Obviously, $\dim\ N_1 = n - 1 \leqslant c_1(p)m - 1 \leqslant c_1(p)(m-1)$, hence we may find i_2 such that $d(e_{i_2}, N_1) \geqslant C_p\alpha_{m-1}$. Iterating this procedure, we can find a sequence $i_1, \dots, i_n$ of distinct indices such that for any $k \leqslant n$ we have $d(e_{i_k}, P_{i_{k-1}} \dots P_{i_1}(N)) \geqslant C_p\alpha_{m-k}$. However, the projections P_{i_j} are contractions, and hence it follows that

$$\begin{aligned} d(e_{i_k}, N) &\geqslant d(P_{i_{k-1}} \dots P_{i_1}(e_{i_k}), P_{i_{k-1}} \dots P_{i_1}(N)) \\ &= d(e_{i_k}, P_{i_{k-1}} \dots P_{i_1}(N)) \\ &\geqslant C_p\alpha_{m-k}, \quad k \leqslant n. \end{aligned}$$

Since $n \leqslant c_1(p)m \leqslant m/2$, we conclude that for $k \leqslant n$ we have

$$\alpha_{m-k} \geqslant \alpha_{m/2} \geqslant \max\{1, 2^{1/p-1/2}\}\alpha_m.$$

Summarizing, we found $n \geqslant (c_p/2)m$ distinct indices i_j such that

$$d(e_{i_j}, N) \geqslant C_p \max\{1, 2^{1/p-1/2}\}\alpha_m$$

for any j. □

Proof of Theorem 2, lower bounds. The lower bounds in the case $p < \alpha$ are now very simple thanks to Corollary 7, part (c), the already proven upper estimate for a_n, and Eq. (2). For $p = \alpha$, we pick some $\tilde{p} < p$ and note that $d_n(X, L_p, q) \geqslant d_n(X, L_{\tilde{p}}, q)$. Hence, we only have to worry about the case $p > \alpha$. Here, we again may employ Corollary 10, and have to find lower bounds for $d_n(\xi, L_p, \alpha)$ only, where $\xi_t := \mathbb{1}_{U \leqslant t}$ with U uniformly distributed on [0, 1]. Using the projection P^k introduced above, we infer that

$$d_n(P^k(\xi), L_p^{(k)}, \alpha) \leqslant d_n(\xi, L_p, \alpha)$$

and further

$$d_n((\tau_k)^{-1}P^k(\xi), \ell_p^k, \alpha) \leqslant k^{1/p} \cdot d_n(\xi, L_p, \alpha).$$

Let e_i be the standard basis in ℓ_p^k, and let us now look at $(\tau_k)^{-1}P^k(\xi)$. The distribution of this random element in ℓ_p^k may be simulated as follows: First, we choose a random variable V, uniformly distributed on [0, 1]; next we choose an independently and uniformly distributed $\imath \in \{1, \dots, k\}$. Then

$$(\tau_k)^{-1}P_k(\xi) \stackrel{d}{=} \eta := V \cdot e_\imath + \sum_{i>\imath} e_i.$$

Hence,

$$d_n(\xi, L_p, \alpha) \geqslant k^{-1/p} \cdot d_n(\eta, \ell_p^k, \alpha). \tag{5}$$

Due to the independence of V and $\imath$, we infer for

$$\hat{\eta} = e_\imath/2 + \sum_{i>\imath} e_i$$

that

$$d_n(\eta, \ell_p^k, \alpha) \geqslant d_n(\hat{\eta}, \ell_p^k, \alpha).$$

Consider now the mapping $\phi : \ell_p^k \to \ell_p^k$, defined by $\phi(e_1) := e_1$, $\phi(e_i) := e_i - e_{i-1}$, $i > 1$. Obviously, $\|\phi\| \leqslant 2$, hence

$$d_n(\phi(\hat{\eta}), \ell_p^k, \alpha) \leqslant 2 d_n(\hat{\eta}, \ell_p^k, \alpha).$$

Note that

$$\phi(\hat{\eta}) = \begin{cases} (e_\imath + e_{\imath+1})/2, & \imath < k, \\ e_\imath/2, & \imath = k. \end{cases}$$

We now choose $k = 2m$ and denote by

$$\pi : \ell_p^{2m} \to \ell_p^m, \quad (x_i)_{i \leqslant 2m} \mapsto (x_{2i})_{i \leqslant m}$$

the projection onto the even coordinates. Then, with $\kappa = \lceil \imath/2 \rceil$ equidistributed on $\{1, \dots, m\}$, we have $\pi(\phi(\hat{\eta})) = e_\kappa/2$, and thus by virtue of (5)

$$d_n(e_\kappa, \ell_p^m, \alpha) \leqslant 4 d_n(\eta, \ell_p^{2m}, \alpha) \leqslant 4(2m)^{1/p} d_n(\xi, L_p, \alpha). \tag{6}$$

The random variable e_κ, at last, is so simple that we can study it directly. For any subspace $N \subseteq \ell_p^m$ we have

$$\left[\mathbb{E}\, d(e_\kappa, N)^\alpha\right]^{1/\alpha} = \left[\frac{1}{m} \sum_{i=1}^m d(e_i, N)^\alpha\right]^{1/\alpha}.$$

Applying Lemma 13, we may, for $\dim N \leqslant c_1(p) m$, estimate

$$\left[\mathbb{E}\, d(e_\kappa, N)^\alpha\right]^{1/\alpha} \geqslant c_2(p)^{1/\alpha} c_3(p) \cdot \max\{1, m^{1/p-1/2}\}.$$

Thus, if we choose, for given $n \in \mathbb{N}$ the value $m \in \mathbb{N}$ such that $(c_1(p)/2) m \leqslant n \leqslant c_1(p) m$, we infer that

$$d_n(e_\kappa, N, \ell_p^m) \geqslant c_4(p) \min\{1, n^{1/p-1/2}\}.$$

From (6) it now follows that

$$d_n(\xi, L_p, \alpha) \geqslant c_5(p) \min\{n^{-1/p}, n^{-1/2}\},$$

which we wanted to prove. □

Appendix A.

Proof of Theorem 6. The proof of Theorem 6 follows the idea of the original Carl inequality; first one establishes estimates for finite-dimensional quantization depending only on dimension and expected norm of the quantized variable, then one decomposes the original random variable into a sum of variable with controllable dimension and expected norm.

For convenience, we introduce non-dyadic quantization numbers,

$$\rho_n(Y, E, q) := \inf\left\{\left(\mathbb{E}\,\|Y - Y_n\|^q\right)^{1/q} \;:\; |\mathrm{supp}\, Y_n| \leqslant n\right\}.$$

Lemma 14. *Let E be a Banach space, $r, q \in (0, \infty)$ with $r < q$, and X a Radon random variable in E with $\mathbb{E}\,\|X\|^q < \infty$ and $\mathrm{rk}\, X \leqslant d$ for some $d \in \mathbb{N}$. Then*

$$\rho_n(X, E, r) \leqslant 6C_{r,q} \cdot \left(\mathbb{E}\,\|X\|^q\right)^{1/q} \cdot n^{-(1-r/q)/d}, \quad n \in \mathbb{N}.$$

Here $C_{r,q}$ depends solely on r, q.

We note that this estimate is not optimal in the order of decay of ρ_n, compare Lemma 6.6 in [8]; however, the constant $C_{r,q}$ is independent of X and d, which surprisingly is more important for us than the correct order.

We shall employ the concept of entropy numbers in the proof of the lemma; for a subset B of a Banach space E, set

$$\varepsilon_n(B) := \inf\left\{\sup_{x\in B} d(x, S) \;:\; S \subseteq E, |S| \leqslant n\right\}.$$

It is well-known (compare, e.g., (1.3.14) of [2]) that for $B = B_{E_0}$ the unit ball of some d-dimensional subspace E_0 we have the estimate

$$\varepsilon_n(B) \leqslant 4n^{-1/d}. \tag{A.1}$$

Proof of Lemma 14. Let $X \in E_0$ a.s., $\dim E_0 \leqslant d$. Set $B := B_{E_0}$, $\beta := -r/q$ and

$$\lambda := \left(\mathbb{E}\,\|X\|^q\right)^{1/q} \cdot \varepsilon_n(B)^{\beta}.$$

Then

$$\varepsilon_n(\lambda B) = \lambda \cdot \varepsilon_n(B) = \left(\mathbb{E}\,\|X\|^q\right)^{1/q} \cdot \varepsilon_n(B)^{1-r/q}.$$

Thus, we know that there exists $S \subseteq E$ such that $|S| \leqslant n$ and

$$\sup_{x\in\lambda B} d(x, S) \leqslant 2\left(\mathbb{E}\,\|X\|^q\right)^{1/q} \cdot \varepsilon_n(B)^{1-r/q}.$$

Let μ be the distribution of X. We take S as a quantization codebook for X and obtain, with $C_r := \max\{1, 2^{1/r-1}\}$, that

$$\begin{aligned}\left(\mathbb{E}\ d(X,S)^r\right)^{\frac{1}{r}} &= \left(\int_E d(x,S)^r\ \mathrm{d}\mu(x)\right)^{\frac{1}{r}}\\ &\leqslant C_r\left[\left(\int_{\lambda B} d(x,S)^r\,\mathrm{d}\mu(x)\right)^{\frac{1}{r}} + \left(\int_{(\lambda B)^c} d(x,S)^r\,\mathrm{d}\mu(x)\right)^{\frac{1}{r}}\right]\\ &\leqslant 2C_r\left[\left(\mathbb{E}\ \|X\|^q\right)^{1/q}\cdot\varepsilon_n(B)^{1-\frac{r}{q}} + \left(\int_E \|x\|^r\cdot\mathbf{I}_{(\lambda B)^c}(x)\ \mathrm{d}\mu(x)\right)^{\frac{1}{r}}\right].\end{aligned}$$

To treat the second summand we note that it equals

$$\left(\mathbb{E}\,\|X\|^r\cdot\mathbf{I}_{(\lambda^r,\infty)}(\|X\|^r)\right)^{\frac{1}{r}} = \left(\int_{\lambda^r}^{\infty}\mathbb{P}\left(\|X\|^r > t\right)\,\mathrm{d}t\right)^{\frac{1}{r}}. \tag{A.2}$$

The Chebyshev inequality allows to estimate

$$\mathbb{P}\left(\|X\|^r > t\right) \leqslant \mathbb{E}\ \|X\|^q\cdot t^{-q/r},$$

and thus (A.2) gives as an upper bound for the second summand the term

$$\left[\mathbb{E}\ \|X\|^q\cdot(1-q/r)^{-1}\left[t^{1-q/r}\right]_{\lambda^r}^{\infty}\right]^{\frac{1}{r}} = C_{r,q}\cdot\left(\mathbb{E}\ \|X\|^q\right)^{1/q}\cdot\varepsilon_n(B)^{\beta(1-q/r)}.$$

Since $\beta(1-q/r) = 1-r/q$, we can continue the estimate of Lemma 14 to

$$\left(\mathbb{E}\ d(X,S)^r\right)^{\frac{1}{r}} \leqslant C_{r,q}\cdot\left(\mathbb{E}\ \|X\|^q\right)^{1/q}\ \varepsilon_n(B)^{1-r/q}.$$

By (A.1), we have $\varepsilon_n(B) \leqslant 6n^{-1/d}$, which implies the assertion. □

Proof of Theorem 6. By standard arguments, we may and will restrict to the case $n = 2^N$. Denote $\alpha_k := k^{-\sigma}$, and set in short

$$S_N := \sup_{j\leqslant N}\alpha_{2^j}^{-1}\cdot d_{2^j}(X,E,q).$$

Since $N \mapsto S_N$ is monotone, we easily see that it suffices to show

$$r_{C_1\cdot 2^N}(X,E,r) \leqslant C_2\cdot\alpha_{2^N}\cdot S_N, \tag{A.3}$$

with C_1, C_2 depending solely on q, r, σ. Let us start by fixing random variables X_j such that $\operatorname{rk} X_j < 2^j$ and

$$\left(\mathbb{E}\ \left\|X - X_j\right\|^q\right)^{1/q} \leqslant 2d_{2^j}(X,E,q), \quad j \leqslant N.$$

Set $X_0 = 0$ and write $Y_j := X_j - X_{j-1}$. Clearly, we have $\operatorname{rk} Y_j < 2^{j+1}$ while $\left(\mathbb{E}\ \|X\|^q\right)^{1/q} \leqslant C_q\cdot d_{2^{j-1}}(X,E,q)$, and X admits a series representation

$$X = \sum_{j=1}^{N} Y_j + (X - X_N). \tag{A.4}$$

In the following we have to separate the cases $r \geqslant 1$ and $r < 1$. Below, we will denote with $C_{r,q,\sigma}$ a constant which may change from line to line, but depends only on r, q, σ.

Case 1. $r \geqslant 1$. In this case, $r_j(X, E, r)$ is additive, i.e., we have $r_{i+j}(X+Y, E, r) \leqslant r_i(X, E, r) + r_j(X, E, r)$. For numbers $n_j \in \mathbb{N}$ to be specified later we infer from (A.4) that

$$r_{\sum_{j=1}^N n_j}(X, E, r) \leqslant \sum_{j=1}^N r_{n_j}(Y_j)_r + \left(\mathbb{E}\, \|X - X_N\|^r\right)^{1/r}. \tag{A.5}$$

We apply estimate (14) on Y_j to derive

$$\begin{aligned} r_{n_j}(Y_j, E, q) &\leqslant C_{r,q} \cdot \left(\mathbb{E}\, \|Y_j\|^q\right)^{1/q} \cdot 2^{-(n_j-1)\cdot(1-r/q)/2^{j+1}} \\ &\leqslant C_{r,q} \cdot d_{2^{j-1}}(X, E, q) \cdot 2^{-(n_j-1)\cdot(1-r/q)/2^{j+1}} \\ &\leqslant C_{r,q,\sigma} \cdot S_N \cdot \alpha_{2^j} \cdot 2^{-(n_j-1)\cdot(1-r/q)/2^{j+1}}. \end{aligned} \tag{A.6}$$

It is time to choose n_j. Set

$$\beta_j := (N - j) + \log_2\left(\frac{\alpha_{2^j}}{\alpha_{2^N}} + 1\right)$$

and define

$$n_j - 1 := 1 + \left\lfloor \frac{2^{j+1} \cdot \beta_j}{(1 - r/q)} \right\rfloor.$$

We may continue estimate (A.6) to

$$\begin{aligned} r_{n_j}(Y_j, E, r) &\leqslant C_{r,q,\sigma} \cdot S_N \cdot \alpha_{2^j} 2^{-\beta_j} \\ &\leqslant C_{r,q,\sigma} \cdot S_N \cdot \alpha_{2^N} 2^{j-N}. \end{aligned}$$

This allows to derive from (A.5) and (A.6) the estimate

$$\begin{aligned} r_{\sum_{j=1}^N n_j}(X, E, r) &\leqslant C_{r,q,\sigma} \cdot S_N \cdot \alpha_{2^N} \cdot \left(\sum_{j=1}^N 2^{j-N} + 1\right) \\ &\leqslant C_{r,q,\sigma} \cdot S_N \cdot \alpha_{2^N}. \end{aligned} \tag{A.7}$$

It remains to estimate the sum over n_i. Since $\alpha_{2^k} = 2^{-\sigma k}$,

$$n_j \leqslant 2 + C_{r,q,\sigma} \cdot 2^{j+1} \cdot (N - j). \tag{A.8}$$

Hence, the sum is bounded by

$$\sum_{j=1}^N n_j \leqslant 2N + C_{r,q,\sigma} \cdot \sum_{j=1}^N 2^{j+1}(N - j) \leqslant C_{r,q,\sigma} \cdot 2^N. \tag{A.9}$$

Together with (A.7), this proves (A.3) in the case $r \geqslant 1$.

Case 2. $r < 1$. Only minor changes have to be made in the argumentation. We use instead of the additivity the r-additivity to derive from (A.4) that

$$r_{\sum_{j=1}^N n_j}(X, E, r)^r \leqslant \sum_{j=1}^n r_{n_j}(Y_j, E, r)^r + \mathbb{E}\, \|X - X_N\|^r. \tag{A.10}$$

We use the same numbers n_j as above, and have the estimate

$$r_{n_j}(Y_j, E, r)^r \leqslant C_{r,q,\sigma} \cdot S_N^r \cdot \alpha_{2^N}^r \cdot 2^{r(j-N)}.$$

Thus, we can conclude from (A.10) that

$$\begin{aligned} r_{\sum_{j=1}^N n_j}(X, E, r)^r &\leqslant C_{r,q,\sigma} \cdot S_N^r \cdot \alpha_{2^N}^r \cdot \left[\sum_{j=1}^N 2^{r(j-N)} + 1 \right] \\ &\leqslant C_{r,q,\sigma} \cdot S_N^r \cdot \alpha_{2^N}^r. \end{aligned}$$

By taking $(1/r)$th power and regarding (A.9), estimate (A.3) follows. □

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